

High Density Effective Theory of QCD

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(Dated: February 1, 2008)

We discuss the salient features of the high density effective theory (HDET) of QCD, elaborating more on the matching for vector-vector correlators and axial-vector-vector correlators, which are related to screening mass and axial anomaly, respectively. We then apply HDET to discuss various color-superconducting phases of dense QCD. We also review a recent proposal to solve the sign problem in dense fermionic matter, using the positivity property of HDET. Positivity of HDET allows us to establish rigorous inequalities in QCD at asymptotic density and to show vector symmetry except the fermion number is not spontaneously broken at asymptotic density.

I. INTRODUCTION

As physics advances, its frontier has expanded. One of the frontiers under active exploration is matter at extreme conditions. Recent surprising data, obtained from heavy-ion collisions and compact stars such as neutron stars, and also some theoretical breakthroughs have stimulated active investigation in this field [1].

How does matter behave as we squeeze it extremely hard? This question is directly related to one of the fundamental questions in Nature; what are the fundamental building blocks of matter and how they interact. According to QCD, matter at high density is quark matter, since quarks interact weaker and weaker as they are put closer and closer.

At what temperature and density does the phase transition to quark matter occur? To determine the phase diagram of thermodynamic QCD is an outstanding problem. The phases of matter are being mapped out by colliding heavy-ions and by observing compact stars. Since QCD has only one intrinsic scale, Λ_{QCD} , the phase transition of QCD matter should occur at that scale as matter is heated up or squeezed down. Indeed, recent lattice QCD calculations found the phase transition does occur at temperature around 175 MeV [2]. Even though lattice QCD has been quite successful at finite temperature but at zero density, it has not made much progress at finite density due to the notorious sign problem. The lattice calculation is usually done in Euclidean space and Euclidean QCD with a chemical potential has a complex measure, which precludes use of importance samplings, the main technique in the Monte Carlo simulation for lattice calculations.

Lattice QCD at finite density is described by a partition function

$$Z(\mu) = \int dA \det(M) e^{-S(A)}, \quad (1)$$

where $M = \gamma_E^\mu D_E^\mu + \mu \gamma_E^4$ is the Dirac operator of Euclidean QCD with a chemical potential μ . The eigenvalues of M are in general complex, since $\gamma_E^\mu D_E^\mu$ is anti-Hermitian while $\mu \gamma_E^4$ is Hermitian. For certain gauge fields such as $A^\mu(-x) = -A^\mu(x)$, M can be mapped into M^\dagger by a similarity transformation and thus its determinant M is nonnegative. However, for generic fields $M \neq P^{-1}M^\dagger P$ and $\det(M)$ is complex.

Recently there have been some progress in lattice simulation at small chemical potential, using a re-weighting method, to find the phase line [3, 4]. Another interesting progress in lattice simulation was made at very high density in [5, 6], where it was shown that for QCD at high density the sign problem is either mild or absent, since the modes, responsible for the complexness of the Dirac determinant, decouple from dynamics or become irrelevant at high baryon density.

II. HIGH DENSITY EFFECTIVE THEORY

At low temperature or energy, most degrees of freedom of quark matter are irrelevant due to Pauli blocking. Only quasi-quarks near the Fermi surface are excited. Therefore, relevant modes for quark matter are quasi-quarks near the Fermi surface and the physical properties of quark matter like the symmetry of the ground state are determined by those modes. High density effective theory (HDET) [7, 8] of QCD is an effective theory for such modes to describe the low-energy dynamics of quark matter.

To find out the modes near the Fermi surface, one needs to know the energy spectrum of QCD, which is very difficult in general since it is equivalent to solving QCD. However, at high density $\mu \gg \Lambda_{\text{QCD}}$, quarks near the Fermi surface carry large momenta and the typical interaction involves a large momentum transfer. Therefore, due to the

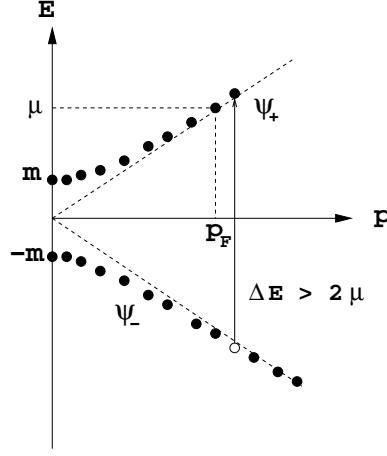


FIG. 1: Energy spectrum of quarks at high density

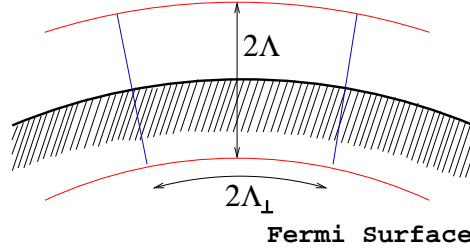


FIG. 2: A patch covering the Fermi surface

asymptotic freedom of QCD, the spectrum near the Fermi surface at high density looks very much like that of free fermion: $(\vec{\alpha} \cdot \vec{p} - \mu + \beta m) \psi_{\pm} = E_{\pm} \psi_{\pm}$, as shown in Fig. 1. We see that at low energy, $E < 2\mu$, the states near the Fermi surface ($|\vec{p}| \simeq p_F$), denoted as ψ_+ , are easily excited while states deep in the Dirac sea, denoted as ψ_- , are hard to excite.

At low energy, the typical momentum transfer by quarks near the Fermi surface is much smaller than the Fermi momentum. Therefore, similarly to the heavy quark effective theory, we may decompose the momentum of quarks near the Fermi surface as

$$p^{\mu} = \mu v^{\mu} + l^{\mu}, \quad (2)$$

where $v^{\mu} = (0, \vec{v}_F)$ and \vec{v}_F is the Fermi velocity. For quark matter, the typical size of the residual momentum is $|l^{\mu}| \sim \Lambda_{\text{QCD}}$, and the Fermi velocity of the quarks does not change for $\mu \gg \Lambda_{\text{QCD}}$, when they are scattered off by soft gluons.

We now introduce patches to cover the Fermi surface, as shown in Fig. 2. The sizes of each patch are 2Λ in vertical direction to the Fermi surface and $2\Lambda_{\perp}$ in horizontal direction. The quarks in a patch are treated to carry a same Fermi velocity.

The energy of the quarks in the patch is given as

$$E = -\mu + \sqrt{p^2 + m^2} = \vec{l} \cdot \vec{v}_F + \frac{l^2}{2\mu} + O\left(\frac{1}{\mu^2}\right). \quad (3)$$

We see that at the leading order in $1/\mu$ expansion, the energy is independent of the residual momentum, \vec{l}_{\perp} , perpendicular to the Fermi velocity. In HDQT, therefore, the perpendicular momentum labels the degeneracy and should satisfy a normalization condition

$$\sum_{\text{patches}} \int_{\Lambda_{\perp}} d^2 l_{\perp} = 4\pi p_F^2. \quad (4)$$

To identify the modes near the Fermi surface, we expand the quark field as

$$\Psi(x) = \sum_{\vec{v}_F} e^{-i\mu \vec{x} \cdot \vec{v}_F} [\psi_+(\vec{v}_F, x) + \psi_-(\vec{v}_F, x)], \quad (5)$$

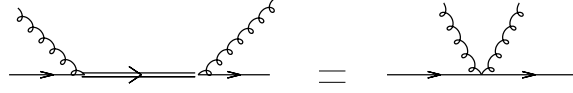


FIG. 3: Tree-level matching: The double line denotes ψ_- modes and the single line ψ_+ .

where $\psi_{\pm}(\vec{v}_F, x)$ satisfies respectively

$$\frac{1 \pm \vec{\alpha} \cdot \hat{v}_F}{2} \psi_{\pm} = \psi_{\pm}. \quad (6)$$

Note that the projection operator $P_{\pm} = (1 \pm \vec{\alpha} \cdot \hat{v}_F)/2$ projects out the particle state, ψ_+ , and the anti-particle state, ψ_- (or more precisely $\bar{\psi}_-$), from the Dirac spinor field Ψ . The quasi-quarks in a patch carries the residual momentum l^{μ} and is given as

$$\psi_+(\vec{v}_F, x) = \frac{1 + \vec{\alpha} \cdot \hat{v}_F}{2} e^{-i\mu\vec{v}_F \cdot \vec{x}} \psi(x) \quad (7)$$

The Lagrangian for quark fields becomes

$$\begin{aligned} \mathcal{L} &= \bar{\Psi} (i\not{D} + \mu\gamma^0) \Psi = \sum_{\vec{v}_F} \bar{\psi} (P_+ + P_-) (\mu V + \not{D}) (P_+ + P_-) \psi \\ &= \bar{\psi}_+ i\not{D}_{\parallel} \psi_+ + \bar{\psi}_- (2\mu\gamma^0 + i\not{D}_{\parallel}) \psi_- + [\bar{\psi}_- i\not{D}_{\perp} \psi_+ + \text{h.c.}], \end{aligned} \quad (8)$$

where we neglected the quark mass term for simplicity and $V^{\mu} = (1, \vec{v}_F)$. The parallel component of the covariant derivative is $D_{\parallel}^{\mu} = V^{\mu} D \cdot V$ and the perpendicular component $D_{\perp} = D - D_{\parallel}$. From the quark Lagrangian one can read off the propagators for $\psi_{\pm}(\vec{v}_F, x)$:

$$S_F^+ = P_+ \frac{i}{\not{D}_{\parallel}}, \quad S_F^- = P_- \frac{i\gamma^0}{2\mu} \left[1 - \frac{i\gamma^0 \not{D}_{\parallel}}{2\mu} + \dots \right]. \quad (9)$$

We see indeed that in HDET the quarks near the Fermi surface or ψ_+ modes are the propagating modes, while ψ_- are not.

By integrating out ψ_- modes and the hard gluons, one obtains the high density effective theory of QCD. In general the integration results in nonlocal terms in the effective theory and one needs to expand them in powers of $1/\mu$. This is usually done by matching the one-light-particle irreducible amplitudes of the microscopic theory with those of the effective theory. For tree-level amplitudes, this is tantamount to eliminating the irrelevant modes, using the equations of motion.

$$\psi_-(\vec{v}_F, x) = -\frac{i\gamma^0}{2\mu + i\not{D}_{\parallel}} \not{D}_{\perp} \psi_+ = -\frac{i\gamma^0}{2\mu} \sum_{n=0}^{\infty} \left(-\frac{i\not{D}_{\parallel}}{2\mu} \right)^n \not{D}_{\perp} \psi_+. \quad (10)$$

For instance, a one-light particle irreducible amplitude in QCD of two gluons and two quarks is matched as

$$\bar{\psi}_+ i\not{D}_{\perp} \psi_-(\vec{v}_F, x) \bar{\psi}_- i\not{D}_{\perp} \psi_+(\vec{v}_F, y) = \bar{\psi}_+ i\not{D}_{\perp} \left(-\frac{i\gamma^0}{2\mu} \right) i\not{D}_{\perp} \psi_+, \quad (11)$$

which is shown in Fig. 3. Similarly one can eliminate the hard gluons. Integrating out hard gluons results in four-Fermi interactions of ψ_+ modes. (See Fig. 4.) One continues matching one-loop or higher-loop amplitudes. One interesting feature of HDET is that a new marginal operator arises at the one-loop matching, when incoming quarks are in Cooper-pairing kinematics, namely when they have opposite Fermi velocities. As shown in Fig. 5, when the incoming quarks have opposite Fermi velocity, the amplitudes in HDET are ultra-violet divergent while QCD amplitudes are not. Therefore, one needs to introduce a four-Fermi operator as a counter term to remove the UV divergence. If we collect all the terms in the effective theory, it has a systematic expansion in $1/\mu$ and coupling constants α_s as

$$\mathcal{L}_{\text{HDET}} = b_1 \bar{\psi}_+ i\gamma_{\parallel}^{\mu} D_{\mu} \psi_+ - \frac{c_1}{2\mu} \bar{\psi}_+ \gamma^0 (\not{D}_{\perp})^2 \psi_+ + \dots, \quad (12)$$

where $b_1 = 1 + O(\alpha_s)$, $c_1 = 1 + O(\alpha_s), \dots$. Note that HDET has a reparametrization invariance, similarly to heavy quark effective theory, which is due to the fact that the Fermi velocity of quarks in a patch is not uniquely

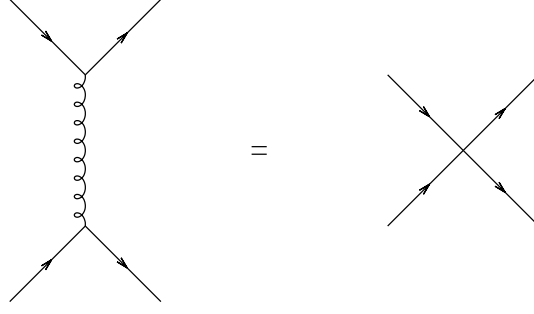


FIG. 4: Tree-level matching: Four-Fermi interaction due to hard gluons

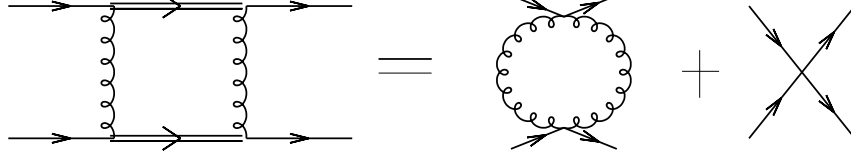


FIG. 5: One-loop matching

determined. For a given quark momentum, the corresponding Fermi velocity is determined up to reparametrization; $\vec{v}_F \rightarrow \vec{v}_F + \delta\vec{l}_\perp/\mu$ and $\vec{l} \rightarrow \vec{l} - \delta\vec{l}$, where $\delta\vec{l}_\perp$ is a residual momentum perpendicular to the Fermi velocity. As in the heavy quark effective theory [9], the renormalization of higher-order operators are restricted due to the reparametrization invariance. For instance, $b_1 = c_1$ at all orders in α_s .

In order for the effective theory to be meaningful, it should have a consistent power-counting. We find the consistent counting in HDET to be for $\Lambda_\perp = \Lambda$

$$\left(\frac{D_\parallel}{\mu}\right)^n \cdot \left(\frac{D_\perp}{\mu}\right)^m \cdot \psi_+^l \sim \left(\frac{\Lambda}{\mu}\right)^{n+m} \Lambda^{3l/2}. \quad (13)$$

To be consistent with the power counting, we impose in loop integration

$$\int_{\Lambda_\perp} d^2 l_\perp l_\perp^n = 0 \quad \text{for } n > 0. \quad (14)$$

So far we have restricted ourselves to operators containing quarks in the same patch. For operators with quarks in different patches, one has to be careful, since the loop integration might jeopardize the power-counting rules. Indeed, consistent counting is to sum up all the hard-loops, as shown by Schäfer [10].

III. MORE ON MATCHING

In HDET, the currents are given in terms of particles and holes but without antiparticles as

$$J^\mu = \sum_{\vec{v}_F} \bar{\psi}(\vec{v}_F, x) \gamma_\parallel^\mu \psi(\vec{v}_F, x) - \frac{1}{2\mu} \psi^\dagger [\gamma_\perp^\mu, i\mathcal{D}_\perp] \psi + \dots, \quad (15)$$

where the color indices are suppressed and we have reverted the notation ψ for ψ_+ henceforth. We find that the HDET current is not conserved unless one adds a counter term. Consider the current correlator

$$\langle J^\mu(x) J^\nu(y) \rangle = \frac{\delta^2 \Gamma_{\text{eff}}}{\delta A_\mu(x) \delta A_\nu(y)} = \int_p e^{-ip \cdot (x-y)} \Pi^{\mu\nu}(p) \quad (16)$$

where the vacuum polarization tensor

$$\Pi_{ab}^{\mu\nu}(p) = -\frac{iM^2}{2} \delta_{ab} \int \frac{d\Omega_{\vec{v}_F}}{4\pi} \left(\frac{-2\vec{p} \cdot \vec{v}_F V^\mu V^\nu}{p \cdot V + i\epsilon \vec{p} \cdot \vec{v}_F} \right) \quad (17)$$

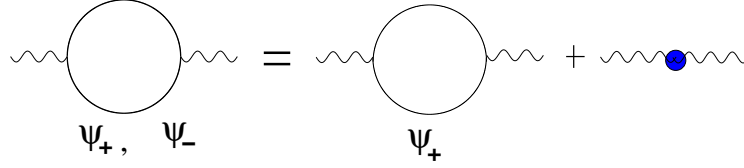


FIG. 6: Matching two-point functions

and $M^2 = N_f g_s^2 \mu^2 / (2\pi^2)$. We see that the vacuum polarization tensor is not transverse, $p_\mu \Pi_{ab}^{\mu\nu}(p) \neq 0$, which means that the current is not conserved. The physical reason for this is that not only modes near the Fermi surface but also modes deep in the Fermi sea respond to external sources collectively. To recover the current conservation in the effective theory, we need to add the Debye screening mass term due to ψ_- (See Fig. 6):

$$\Gamma^{\text{eff}} \mapsto \tilde{\Gamma}^{\text{eff}} = \Gamma^{\text{eff}} - \int_x \frac{M^2}{2} \sum_{\vec{v}_F} A_\mu A_\nu g_\perp^{\mu\nu}. \quad (18)$$

Then vacuum polarization tensor becomes

$$\Pi^{\mu\nu}(p) \mapsto \tilde{\Pi}^{\mu\nu}(p) = \Pi^{\mu\nu} - \frac{i}{2} \sum_{\vec{v}_F} g_\perp^{\mu\nu} M^2. \quad (19)$$

The modified polarization tensor is now transverse, $p_\mu \tilde{\Pi}^{\mu\nu} = 0$.

Now, let us consider the divergence of axial currents in HDET, which is related to the axial anomaly and also to how the quark matter responds to external axial-current sources like electroweak probes.

It is easy to show that the axial anomaly in dense matter is independent of density or the chemical potential μ [11]. In general one may re-write the divergence of axial currents in dense QCD as follows:

$$\langle \partial_\mu J_5^\mu \rangle = \frac{g^2}{16\pi^2} \tilde{F}_{\mu\alpha} F^{\mu\alpha} + \Delta^{\alpha\beta}(\mu) A_\alpha A_\beta, \quad (20)$$

where the first term is the usual axial anomaly in vacuum and the second term is due to matter. However, one can explicitly calculate the second term, which is finite, to find $\Delta^{\alpha\beta}(\mu) = 0$. In HDET, the axial anomaly due to modes near the Fermi surface is given as

$$\sum_{\vec{v}_F} \int_{x,y} e^{ik_1 \cdot x + ik_2 \cdot y} \langle \partial_\mu J_5^\mu(\vec{v}_F, 0) J^\alpha(\vec{v}_F, x) J^\beta(\vec{v}_F, y) \rangle \equiv \Delta_{\text{eff}}^{\alpha\beta} \quad (21)$$

By explicit calculation we find

$$\Delta_{\text{eff}}^{0i} = -\frac{g^2}{4\pi^2} \cdot \frac{1}{3} (\vec{k}_1 \times \vec{k}_2)^i, \quad \Delta_{\text{eff}}^{ij} = \frac{g^2}{4\pi^2} \frac{2}{3} \epsilon^{ijl} (k_{10} k_{2l} - k_{1l} k_{20}). \quad (22)$$

We see that the modes near the Fermi surface contributes only some parts of the axial anomaly. As in the vector current, the rest should come from modes in the deep Fermi sea and from anti-particles. To recover the full axial anomaly we add a counter term (See Fig. 7), which is two thirds of the axial anomaly plus a Chern-Simons term:

$$\tilde{\Delta}_{\text{eff}}^{\alpha\beta} = \Delta_{\text{eff}}^{\alpha\beta} + \frac{g^2}{6\pi^2} \epsilon^{\alpha\beta\rho\sigma} k_{1\rho} k_{2\sigma} + \frac{g^2}{12\pi^2} \epsilon^{\alpha\beta 0l} (k_{10} k_{2l} - k_{1l} k_{20}). \quad (23)$$

IV. COLOR SUPERCONDUCTIVITY IN DENSE QCD

At high density, quarks in dense matter interact weakly with each other and form a Fermi sea, due to asymptotic freedom. When the energy is much less than the quark chemical potential ($E \ll \mu$), only the quarks near the Fermi surface are relevant. The dynamics of quarks near the Fermi surface is effectively one-dimensional, since excitations along the Fermi surface do not cost any energy. The momentum perpendicular to the Fermi momentum just labels

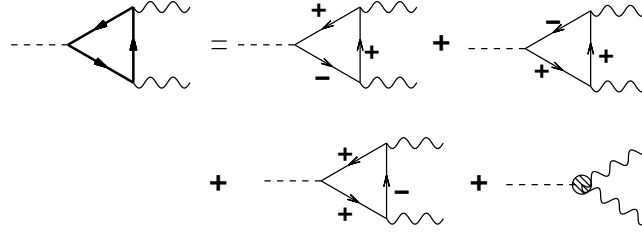


FIG. 7: Matching axial anomaly. + denotes ψ_+ and - denotes ψ_- .

the degeneracy, similarly to the perpendicular momentum of charged particle under external magnetic field. This dimensional reduction due to the presence of Fermi surface makes possible for quarks to form a Cooper pair for any arbitrary weak attraction, since the critical coupling for the condensation in (1+1) dimensions is zero, known as the Cooper theorem in condensed matter.

While, in the BCS theory, such attractive force for electron Cooper pair is provided by phonons, for dense quark matter, where phonons are absent, the gluon exchange interaction provides the attraction, as one-gluon exchange interaction is attractive in the color anti-triplet channel[31]. One therefore expects that color anti-triplet Cooper pairs will form and quark matter is color superconducting, which is indeed shown more than 20 years ago [13, 14, 15].

At intermediate density, quarks and gluons are strongly interacting and gluons are therefore presumably screened. Then, QCD at intermediate density may be modelled by four-Fermi interactions and higher-order terms by massive gluons.

$$\mathcal{L}_{\text{QCD}}^{\text{eff}} \ni \frac{G}{2} \bar{\psi}\psi\bar{\psi}\psi + \dots, \quad (24)$$

where the ellipsis denotes higher-order terms induced by massive gluons. When the incoming quarks have opposite momenta, the four-Fermi interaction is marginally relevant, if attractive, and all others are irrelevant. As the renormalization group flows toward the Fermi surface, the attractive four-Fermi interaction is dominant and blows up, resulting in a Landau pole, which can be avoided only when a gap opens at the Fermi surface. This is precisely the Cooper-instability of the Fermi surface. The size of gap can be calculated by solving the gap equation, which is derived by the variational principle that the gap minimizes the vacuum energy:

$$0 = \frac{\partial V_{\text{BCS}}(\Delta)}{\partial \Delta} = \frac{\Delta}{G} - i \int \frac{d^4 k}{(2\pi)^4} \frac{\Delta}{k_0^2 - (\vec{k} \cdot \vec{v}_F)^2 - \Delta^2}, \quad (25)$$

which gives

$$\Delta = -iG \int \frac{d^4 k}{(2\pi)^4} \frac{\Delta}{[(1+i\epsilon)k_0]^2 - (\vec{k} \cdot \vec{v}_F)^2 - \Delta^2}. \quad (26)$$

We note that the integrand in Eq. 26 does not depend on k_\perp , whose integration gives the density of states at the Fermi surface, and the $i\epsilon$ prescription is consistent with the Feynman propagator. The pole occurs at

$$k_0 = \pm \sqrt{(\vec{k} \cdot \vec{v}_F)^2 + \Delta^2} \mp i\epsilon \quad (27)$$

or in terms of full momentum $p = \mu v + k$ it occurs at

$$p_0 = \pm \sqrt{(|\vec{p}| - \mu)^2 + \Delta^2} \mp i\epsilon. \quad (28)$$

We find the solution to the gap equation

$$\Delta_0 = 2\mu \exp\left(-\frac{\pi^2}{2G\mu^2}\right). \quad (29)$$

For generic parameters of dense QCD, the gap is estimated to be $10 \sim 100$ MeV at the intermediate density. The free energy of the BCS state is given as

$$\begin{aligned} V_{\text{BCS}}(\Delta_0) &= \int_0^{\Delta_0} \frac{\partial V_{\text{BCS}}}{\partial \Delta} d\Delta \\ &= \frac{4\mu^2}{G} \int_0^{x_0} (x + g^2 \ln x) dx = -\frac{\mu^2}{4\pi^2} \Delta_0^2, \end{aligned} \quad (30)$$

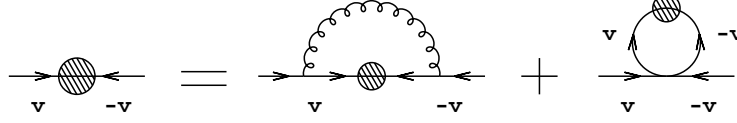


FIG. 8: Eliashberg equation at high density.

where $x = \Delta/(2\mu)$ and $g^2 = 2G\mu^2/\pi^2$. At high density magnetic gluons are not screened though electric gluons are screened [16, 17, 18]. The long-range pairing force mediated by magnetic gluons leads to the Eliashberg gap equation (See Fig. 8).

$$\Delta(p_0) = \frac{g_s^2}{36\pi^2} \int_{-\mu}^{\mu} dq_0 \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta^2}} \ln \left(\frac{\bar{\Lambda}}{|p_0 - q_0|} \right), \quad (31)$$

where $\bar{\Lambda} = 4\mu/\pi \cdot (\mu/M)^5 e^{3/2\xi}$ and ξ is a gauge parameter. Due to the unscreened but Landau-damped gluons, there is an extra (infrared) logarithmic divergence in the gap equation, when the incoming quark momentum is collinear with the gluon momentum. The Cooper-pair gap at high density is found to be [18, 19]

$$\Delta_0 = \frac{2^7 \pi^4}{N_f^{5/2}} e^{3\xi/2+1} \cdot \frac{\mu}{g_s^5} \exp \left(-\frac{3\pi^2}{\sqrt{2}g_s} \right). \quad (32)$$

The numerical prefactor of the gap is not complete, since the contributions from subleading corrections to the gap equation that lead to logarithmic divergences, such as the wavefunction renormalization and the vertex corrections, are not taken into account. Recently, however, the contributions to the prefactor, coming from the vertex corrections and the wavefunction renormalization for quarks were calculated by finding a (nonlocal) gauge [20], where the quark wavefunction is not renormalized for all momenta, $Z(p) = 1$. At the nonlocal gauge, $\xi \simeq 1/3$. The subleading corrections therefore increase the leading-order gap at the Coulomb gauge by about two thirds.

V. QUARK MATTER UNDER STRESS

It is quite likely to find dense quark matter inside compact stars like neutron stars. However, when we study the quark matter in compact stars, we need to take into account not only the charge and color neutrality of compact stars and but also the mass of the strange quark, which is not negligible at the intermediate density. By the neutrality condition and the strange quark mass, the quarks with different quantum numbers in general have different chemical potentials and different Fermi momenta. When the difference in the chemical potential becomes too large the Cooper-pairs breaks or other exotic phases like kaon condensation or crystalline phase is more preferred to the BCS phase.

Let us consider for example the pairing between up and strange quarks in chemical equilibrium. The energy spectrum of up quarks is given as

$$E = -\mu \pm |\vec{p}|, \quad (33)$$

while the energy of strange quarks of mass M_s becomes

$$E = -\mu \pm \sqrt{|\vec{p}|^2 + M_s^2}. \quad (34)$$

The Fermi sea of up and strange quarks is shown in Fig. 9. Because of the strange quarks mass, they have different Fermi momenta. Note that the Cooper-pairing occurs for quarks with same but opposite momenta. Therefore, at least one of the pairing quarks should be excited away from the Fermi surface, costing some energy. Let us suppose that the Cooper-pair gap opens at $|\vec{p}| = \bar{p}$ between two Fermi surfaces, $p_F^s \leq \bar{p} \leq p_F^u$.

To describe such pairing, we consider small fluctuations of up and strange quarks near \bar{p} . The energy of such fluctuations of up and down quarks is respectively

$$E_u = -\mu + |\bar{p} + \vec{l}| \simeq -\delta\mu^u + \vec{v}_u \cdot \vec{l}, \quad E_s \simeq -\delta\mu^s + \vec{v}_s \cdot \vec{l}, \quad (35)$$

where $\delta\mu^u = \mu - \bar{p}$ and $\delta\mu^s = \mu - \sqrt{\bar{p}^2 + M_s^2}$. \vec{v}_u and \vec{v}_s are the velocities of up and strange quarks at $|\vec{p}| = \bar{p}$. Let Δ be the BCS gap for the u, s pairing. Then, the Lagrangian for the u, s quarks is given as

$$\mathcal{L} = \bar{u} (i \not{\partial} + \mu\gamma^0) u + \bar{s}_c (i \not{\partial} - \mu\gamma^0 - M_s) s_c - \Delta \bar{s}_c u + \text{h.c.} + \mathcal{L}_{\text{int}}, \quad (36)$$

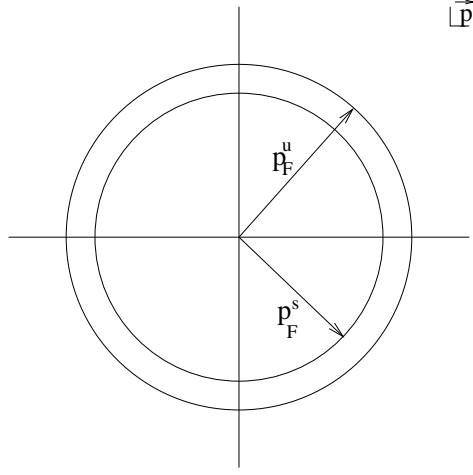


FIG. 9: Fermi sea of up and strange quarks.

where s_c is the charge conjugate field of s quark. In HDET, the Lagrangian becomes

$$\mathcal{L}_{\text{HDET}} \ni u^\dagger (iV_u \cdot \partial + \delta\mu^u) u + s_c^\dagger (i\bar{V}_s \cdot \partial - \delta\mu^s) s_c - \Delta \bar{s}_c u + \text{h.c.}, \quad (37)$$

where $V_u = (1, \vec{v}_u)$ and $\bar{V}_s = (1, -\vec{v}_s)$. The Cooper-pair gap equation is then

$$\Delta(p) = \int_l \frac{i\Delta(l) K(p-l)}{\left[(1+i\epsilon)l_0 - \vec{l} \cdot \vec{v}_u + \delta\mu^u\right] \left[(1+i\epsilon)l_0 + \vec{l} \cdot \vec{v}_s - \delta\mu^u\right] - \Delta^2}, \quad (38)$$

where K is the kernel for the gap equation and is a constant for the four-Fermi interaction. By examining the pole structure, we see that the Cooper-pair gap does not exist when

$$-\delta\mu^u \delta\mu^s > \frac{\Delta^2}{4}. \quad (39)$$

Only when $-\delta\mu^u \delta\mu^s < \Delta^2/4$, one can shift $l_0 \rightarrow l'_0 = l_0 + \delta\mu^u$ or $l_0 \rightarrow l'_0 = l_0 - \delta\mu^s$ without altering the pole structure. Note that the gap becomes biggest when $\delta\mu^u = -\delta\mu^s (\equiv \delta\mu)$, which determines the pairing momentum to be

$$\bar{p} = \mu - \frac{M_s^2}{4\mu}. \quad (40)$$

If $\delta\mu < \Delta/2$ or $\Delta > M_s^2/(2\mu)$, the solution to the Cooper-pair gap exists. The gap equation then can be written as, shifting l_0 , in Euclidean space

$$\Delta(p) = \int \frac{d^4l}{(2\pi)^4} \frac{\Delta(l)}{l_\parallel^2 + \Delta^2} K(l-p), \quad (41)$$

where $l_\parallel^2 = l_0^2 + c^2(\vec{l} \cdot \hat{v})^2$ and $c^2 = \bar{p}/\sqrt{\bar{p}^2 + M_s^2}$. In HDET, one can easily see that the Cooper-pair gap closes if the effective chemical potential difference, $2\delta\mu$, due to an external stress, exceeds the Cooper-pair gap when there is no stress. One should note that even before the Cooper-pair gap closes other gap may open as shown by many authors [21, 22]. But, one needs to compare the free energy of each phases to find the true ground state for quark matter under stress.

VI. POSITIVITY OF HDET

Fermionic dense matter generically suffers from the sign problem, which has thus far precluded lattice simulations [23]. However, the sign problem usually associated with fermions is absent if one considers only low-energy degrees of freedom. The complexity of the measure of fermionic dense matter can be ascribed to modes far from

the Fermi surface, which are irrelevant to dynamics at sufficiently high density in most cases, including quark matter [5, 6]. For modes near the Fermi surface, there is a discrete symmetry, relating particles and holes, which pairs the eigenvalues of the Dirac operator to make its determinant real and nonnegative. Especially, the low energy effective theory of dense QCD has positive Euclidean path integral measure, which allows one to establish rigorous inequalities that the color-flavor locked (CFL) phase is the true vacuum of three flavor, massless QCD.

As simple example, let us consider a fermionic matter in 1+1 dimensions, where non-relativistic fermions are interacting with a gauge field A . The action is in general given as

$$S = \int d\tau dx \psi^\dagger [(-\partial_\tau + i\phi + \epsilon_F) - \epsilon(-i\partial_x + A)] \psi, \quad (42)$$

where $\epsilon(p) \simeq p^2/(2m) + \dots$ is the energy as a function of momentum. Low energy modes have momentum near the Fermi points and have energy, measured from the Fermi points,

$$E(p \pm p_F) \simeq \pm v_F p, \quad v_F = \left. \frac{\partial E}{\partial p} \right|_{p_F}. \quad (43)$$

If the gauge fields have small amplitude and are slowly varying relative to scale p_F , the fast modes are decoupled from low energy physics. The low energy effective theory involving quasi particles and gauge fields has a positive, semi-definite determinant.

To construct the low energy effective theory of the fermionic system, we rewrite the fermion fields as

$$\psi(x, \tau) = \psi_L(x, \tau) e^{+ip_F x} + \psi_R(x, \tau) e^{-ip_F x}, \quad (44)$$

where $\psi_{L,R}$ describes the small fluctuations of quasiparticles near the Fermi points. Using $e^{\pm ip_F x} E(-i\partial_x + A) e^{\mp ip_F x} \psi(x) \approx \pm v_F (-i\partial_x + A) \psi(x)$, we obtain

$$S_{\text{eff}} = \int_{\tau, x} \left[\psi_L^\dagger (-\partial_\tau + i\phi + i\partial_x - A) \psi_L + \psi_R^\dagger (-\partial_\tau + i\phi - i\partial_x + A) \psi_R \right]. \quad (45)$$

Introducing the Euclidean (1+1) gamma matrices $\gamma_{0,1,2}$ and $\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_2)\psi$, we obtain a positive action:

$$S_{\text{eff}} = \int d\tau dx \bar{\psi} \gamma^\mu (\partial_\mu + iA_\mu) \psi \equiv \int d\tau dx \bar{\psi} \not{D} \psi. \quad (46)$$

Since $\not{D} = \gamma_2 \not{D}^\dagger \gamma_2$, the determinant of \not{D} is positive, semi-definite.

In this example, we see that near the Fermi surface, modes have low energy, slowly varying, and thus lead to an effective theory without any sign problem what so ever if they couple to slowly varying background fields. QCD at high baryon density falls into this category, since the coupling constant is small at high energy due to asymptotic freedom.

HEDT of quark matter is described by

$$\mathcal{L}_{\text{HDET}} = \bar{\psi}_+ i\gamma_\parallel^\mu D_\mu \psi_+ - \frac{1}{2\mu} \bar{\psi}_+ \gamma^0 (\not{D}_\perp)^2 \psi_+ + \dots, \quad (47)$$

where $\gamma_\parallel^\mu = (\gamma^0, \vec{v}_F \vec{v}_F \cdot \vec{\gamma}) = \gamma^\mu - \gamma_\perp^\mu$. We see that the leading term has a positive determinant, since

$$M_{\text{eff}} = \gamma_\parallel^E \cdot D(A) = \gamma_5 M_{\text{eff}}^\dagger \gamma_5. \quad (48)$$

In order to implement this HDET on lattice, it is convenient to introduce an operator formalism, where the velocity is realized as an operator,

$$\vec{v} = \frac{-i}{\sqrt{-\nabla^2}} \frac{\partial}{\partial \vec{x}}. \quad (49)$$

Then the quasi quarks near the Fermi surface become

$$\psi = \exp(+i\mu x \cdot v) \frac{1 + \alpha \cdot v}{2} \psi_+. \quad (50)$$

Now, neglecting the higher order terms, the Lagrangian becomes with $X = \exp(i\mu x \cdot v)(1 + \alpha \cdot v)/2$,

$$\mathcal{L}_{\text{HDET}} = \bar{\psi}_+ \gamma_\parallel^\mu (\partial^\mu + iA_+^\mu) \psi_+, \quad (51)$$

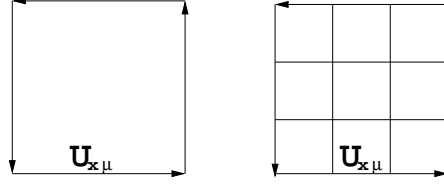


FIG. 10: Simulation with two lattices with different lattice spacings

where $A_+^\mu = X^\dagger A^\mu X$ denotes soft gluons whose momentum $|p_\mu| < \mu$. Since $v \cdot \partial v \cdot \gamma = \partial \cdot \gamma$, we get

$$\gamma_\parallel^\mu \partial^\mu = \gamma^\mu \partial^\mu \quad (52)$$

which shows that the operator formalism automatically covers modes near the full Fermi surface.

Integrating out the fast modes, modes far from the Fermi surface and hard gluons, the QCD partition function (1) becomes

$$Z(\mu) = \int dA_+ \det(M_{\text{eff}}) e^{-S_{\text{eff}}(A_+)}, \quad (53)$$

where

$$S_{\text{eff}} = \int_{x_E} \left(\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{M^2}{16\pi} A_{\perp\mu}^a A_{\perp\mu}^a \right) + \dots \quad (54)$$

and $A_\perp = A - A_\parallel$, the Debye mass $M = \sqrt{N_f/(2\pi^2)} g_s \mu$. At high density the higher order terms ($\sim \Lambda/\mu$) are negligible and the effective action becomes positive, semi-definite. Therefore, though it has non-local operators, HDET in the operator formalism, free from the sign problem, can be used to simulate the Fermi surface physics like superconductivity. Furthermore, being exactly positive at asymptotic density, HDET allows to establish rigorous inequalities relating bound state masses and forbidding the breaking of vector symmetries, except baryon number, in dense QCD [6].

With the help of previous two examples, we propose a new way of simulating dense QCD, which evades the sign problem. Integrating out quarks far from the Fermi surface, which are suppressed by $1/\mu$ at high density, we can expand the determinant of Dirac operator at finite density,

$$\det(M) = [\text{real, positive}] \left[1 + \mathcal{O}\left(\frac{\mathbf{F}}{\mu^2}\right) \right]. \quad (55)$$

As long as the gauge fields are slowly varying, compared to the chemical potential μ , the sign problem can be evaded. As a solution to the sign problem, we propose to use two lattices with different spacings, a finer lattice with a lattice spacing $a_{\text{det}} \sim \mu^{-1}$ for fermions and a coarser lattice with a lattice spacing $a_{\text{gauge}} \ll \mu^{-1}$ for gauge fields and then compute the determinant on such lattices.

The determinant is a function of plaquettes $\{\mathbf{U}_{\mathbf{x}\mu}\}$ which are obtained by interpolation from the plaquettes on the coarser lattice of lattice spacing a_{gauge} . To get the link variables for the finer lattice, we interpolate the link variables $\mathbf{U}_{\mathbf{x}\mu} \in SU(3)$ (see Fig. 10): Connect any two points g_1, g_2 on the group manifold as

$$g(t) = g_1 + t(g_2 - g_1), \quad 0 \leq t \leq 1. \quad (56)$$

For importance sampling in the lattice simulation, one can use the leading part of the determinant, $[\text{real, positive}]$. This proposal provides a nontrivial check on analytic results at asymptotic density and can be used to extrapolate to intermediate density. Furthermore, it can be applied to condensed matter systems like High- T_c superconductors, which in general suffers from a sign problem.

Positivity of the measure allows for rigorous QCD inequalities at asymptotic density. For example, inequalities among masses of bound states can be obtained using bounds on bare quasiparticle propagators. One subtlety that arises is that a quark mass term does not lead to a quasiparticle gap (the mass term just shifts the Fermi surface). Hence, for technical reasons the proof of non-breaking of vector symmetries [24] must be modified. (Naive application of the Vafa-Witten theorem would preclude the breaking of baryon number that is observed in the color-flavor-locked (CFL) phase [25]). A quasiparticle gap can be inserted by hand to regulate the bare propagator, but it will explicitly violate baryon number. However, following the logic of the Vafa-Witten proof, any symmetries which are preserved

by the regulator gap cannot be broken spontaneously. One can, for example, still conclude that isospin symmetry is never spontaneously broken (although see below for a related subtlety). In the case of three flavors, one can introduce a regulator d with the color and flavor structure of the CFL gap to show rigorously that none of the symmetries of the CFL phase are broken at asymptotic density. On the other hand, by applying anomaly matching conditions [26], we can prove that the $SU(3)_A$ symmetries *are* broken. We therefore conclude that the CFL phase is the true ground state for three light flavors at asymptotic density, a result that was first established by explicit calculation [8, 27, 28].

To examine the long-distance behavior of the vector current, we note that the correlator of the vector current for a given gauge field A can be written as

$$\langle J_\mu^a(\vec{v}_F, x) J_\nu^b(\vec{v}_F, y) \rangle^A = -\text{Tr} \gamma_\mu T^a S^A(x, y; d) \gamma_\nu T^b S^A(y, x; d), \quad (57)$$

where the $SU(N_f)$ flavor current $J_\mu^a(\vec{v}_F, x) = \bar{\psi}_+(\vec{v}_F, x) \gamma_\mu T^a \psi_+(\vec{v}_F, x)$. The propagator with $SU(3)_V$ -invariant IR regulator d is given as

$$S^A(x, y; d) = \langle x | \frac{1}{M} | y \rangle = \int_0^\infty d\tau \langle x | e^{-i\tau(-iM)} | y \rangle$$

where with $D = \partial + iA$

$$M = \gamma_0 \begin{pmatrix} D \cdot V & d \\ d^\dagger & D \cdot \bar{V} \end{pmatrix} \quad (58)$$

Since the eigenvalues of M are bounded from below by d , we have

$$\left| \langle x | \frac{1}{M} | y \rangle \right| \leq \int_R^\infty d\tau e^{-d\tau} \sqrt{\langle x | x \rangle} \sqrt{\langle y | y \rangle} = \frac{e^{-dR}}{d} \sqrt{\langle x | x \rangle} \sqrt{\langle y | y \rangle}, \quad (59)$$

where $R \equiv |x - y|$. The current correlators fall off rapidly as $R \rightarrow \infty$;

$$\begin{aligned} & \left| \int dA_+ \det M_{\text{eff}}(A) e^{-S_{\text{eff}}} \langle J_\mu^A(\vec{v}_F, x) J_\nu^B(\vec{v}_F, y) \rangle^{A_+} \right| \\ & \leq \int_{A_+} \left| \langle J_\mu^A(\vec{v}_F, x) J_\nu^B(\vec{v}_F, y) \rangle^{A_+} \right| \leq \frac{e^{-2dR}}{d^2} \int_{A_+} |\langle x | x \rangle| |\langle y | y \rangle|, \end{aligned} \quad (60)$$

where we used the Schwartz inequality in the first inequality, since the measure of the effective theory is now positive, and equation (59) in the second inequality. The IR regulated vector currents do not create massless modes out of the vacuum or Fermi sea, which implies that there is no Nambu-Goldstone mode in the $SU(3)_V$ channel. Therefore, for three massless flavors $SU(3)_V$ has to be unbroken as in CFL. The rigorous result provides a non-trivial check on explicit calculations, and applies to any system in which the quasiparticle dynamics have positive measure.

It is important to note the order of limits necessary to obtain the above results. Because there are higher-order corrections to the HDET, suppressed by powers of Λ/μ , that spoil its positivity, there may be contributions on the RHS of (60) of the form

$$\mathcal{O}\left(\frac{\Lambda}{\mu}\right) f(R), \quad (61)$$

where $f(R)$ falls off more slowly than the exponential in (60). To obtain the desired result, we must first take the limit $\mu \rightarrow \infty$ at fixed Λ before taking $R \rightarrow \infty$. Therefore, our results only apply in the limit of asymptotic density.

Although our result precludes breaking of vector symmetries at asymptotic density in the case of three *exactly* massless quarks [30], it does not necessarily apply to the case when the quark masses are allowed to be slightly non-zero. In that case the results depend on precisely how the limits of zero quark masses and asymptotic density are taken, as we discuss below.

In [22] the authors investigate the effect of quark masses on the CFL phase. These calculations are done in the asymptotic limit, and are reliable for sufficiently small quark masses. When $m_u = m_d \equiv m \ll m_s$ (unbroken $SU(2)$ isospin, but explicitly broken $SU(3)$), one finds a kaon condensate. The critical value of m_s at which the condensate forms is $m_s^* \sim m^{1/3} \Delta_0^{2/3}$, where Δ_0 is the CFL gap (see, in particular, equation (8) of the first paper). As kaons transform as a doublet under isospin, the vector $SU(2)$ symmetry is broken in seeming contradiction with our result.

However, a subtle order of limits is at work here. For simplicity, let us set $m = 0$. Note that the CFL regulator d , which was inserted by hand, explicitly breaks $SU(3)_A$ through color-flavor locking, leading to small positive mass squared for the pions and kaons, given as

$$m_{\pi, K}^2 \sim \alpha_s d^2 \ln\left(\frac{\mu}{d}\right). \quad (62)$$

The meson mass is not suppressed by $1/\mu$, since, unlike the Dirac mass term, the regulator, being a Majorana mass, does not involve antiquarks [29].

Therefore, even when the light quarks are massless, there is a critical value of m_s necessary to drive negative the mass-squared of kaons and cause condensation:

$$m_s^* \sim \left[g_s d \mu \ln \left(\frac{\mu}{d} \right) \right]^{1/2} > (d\mu)^{1/2} , \quad (63)$$

where g_s is the strong coupling constant. Note the product of g_s with the logarithm grows as μ gets large. To obtain our inequality we must keep the regulator d non-zero until the end of the calculation in order to see the exponential fall off. To find the phase with kaon condensation identified in [22] we must keep m_s larger than m_s^* . (Note $\mu \rightarrow \infty$, so to have any chance of finding this phase we must take $d \rightarrow 0$ keeping dR large and $d\mu$ small.)

Since the UV cutoff of the HDET must be larger than m_s , we have

$$1 > \left(\frac{m_s^*}{\Lambda} \right)^2 > \frac{d}{\Lambda} \frac{\mu}{\Lambda} , \quad (64)$$

which implies

$$\frac{\Lambda}{\mu} f(R) > \frac{d}{\Lambda} f(R) . \quad (65)$$

Note the right hand side of this inequality does not necessarily fall off at large R , and also does not go to zero for $\mu \rightarrow \infty$ at fixed Λ and d . This is a problem since to apply our inequality the exponential falloff from (60) must dominate the correction term (61), which is just the left hand side of (65). Combining these equations, we see that the exponential falloff of the correlator is bounded below,

$$\frac{e^{-2dR}}{d^2} > \frac{d}{\Lambda} f(R) , \quad (66)$$

in the scaling region with a kaon condensate, $m_s > m_s^*$.

Alternatively, if we had taken m_s to be finite for fixed regulator d (so that, as $\mu \rightarrow \infty$, eventually $m_s < m_s^*$), the inequality in (60) could be applied to exclude a Nambu-Goldstone boson, but we would find ourselves in the phase without a kaon condensate.

Acknowledgements

The author wishes to thank Mark Alford, Phillipe de Forcrand, Simon Hands, Krishna Rajagopal, Francesco Sannino, Thomas Schäfer for useful discussions. The author is thankful especially to Steve Hsu for the critical discussions and for the collaboration, upon which some of this lecture is based. This work is supported by KOSEF grant number R01-1999-000-00017-0.

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 - [30] To investigate spontaneous symmetry breaking, one ordinarily has to start at finite volume and insert a source which explicitly breaks the symmetry. The source is removed only after the infinite volume limit is taken. We stress that the source does not have to be a quark mass (it could be a higher dimension operator), so one can investigate symmetry breaking even when the quark mass is exactly zero throughout the calculation. (To be precise, a quark mass does not explicitly violate vector symmetries, so it cannot play the role of the source in the thermodynamic limit needed here.)
 - [31] There is also an attractive force between quarks and holes in the color octet channel: $\langle \bar{\psi}_i(-\vec{p})\psi_j(\vec{p}) \rangle \neq 0$, which corresponds to a density wave. However, because of the momentum conservation, the density wave condensate does not enjoy the full Fermi surface degeneracy. Indeed, for QCD, the diquark condensate is energetically preferred to the density wave condensate [12].